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Approximate Solutions to the Zakharov Equations via Finite Differences*

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An energy-preserving, linearly implicit finite-difference scheme is presented for computing solutions to the periodic initial-value problem for the Zakharov equations. Solitary waves and colliding solitary waves are computed, and a comparison is made with previous calculations.

I. INTRODUCTION

Zakharov introduced in [10] a system of equations to model the propagation of Langmuir waves in a plasma. The fluid-type equations take the form

$$iE_t + E_{xx} = NE \qquad \text{(ZS.E)}$$

$$N_{tt} - N_{xx} = \frac{\partial^2}{\partial x^2}(|E|^2). \qquad \text{(ZS.N)}$$

Here E is the envelope of the high-frequency electric field, and N is the deviation of the ion density from its equilibrium value.

We study here the periodic initial-value problem for a system with period L. Smooth initial values are prescribed for $0 \leqslant x \leqslant L$:

$$E(x, 0) = E^0(x); \qquad N(x, 0) = N^0(x), \qquad N_t(x, 0) = N^1(x). \qquad (1)$$

We know of only one previous study [6] of this system. There a spectral method is used; solitary waves and the interaction of two colliding solitary waves are computed. Although the convergence of the algorithm in [6] has not been demonstrated, computational studies of errors in [6] seem convincing.

One purpose of the present paper is to introduce a new finite-difference scheme for (ZS). This scheme preserves discrete versions of the two standard invariants for (ZS):

$$\int_0^L |E(x, t)|^2 \, dx = \text{const.} \qquad (2)$$

$$\int_0^L (|E_x|^2 + \tfrac{1}{2}(v^2 + N^2) + N|E|^2) \, dx = \text{const.} \qquad (3)$$

We may call the expression in (3) the "energy"; there, v is defined by

$$v = -u_x, \; u_{xx} = N_t. \qquad (4)$$

In [3] we have proven that this scheme is first-order convergent in a natural "energy norm" (defined below) to the exact solution.

The second purpose of the present paper is to confirm the computational experiments from [6] involving the collision of two oppositely-directed solitary waves.

II. THE FINITE-DIFFERENCE SCHEME

Denote by L the period of the system, and let $T > 0$ be an arbitrary final time. Given a positive integer J, we put

$$\Delta x = \frac{L}{J}; \qquad x_j = j\,\Delta x \qquad \text{for} \quad j = 0, ..., J. \qquad (5)$$

For $\Delta t > 0$ and an integer $n > 0$ with $n\,\Delta t \leqslant T$, we put

$$t^k = k\,\Delta t \qquad \text{for} \quad k = 0, ..., n. \qquad (6)$$

The standard difference operators are

$$\delta u_k = \Delta x^{-1}(u_{k+1} - u_k) \qquad (7)$$

$$\delta^2 u_k = \Delta x^{-2}(u_{k+1} - 2u_k + u_{k-1}). \qquad (8)$$

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The scheme can then be written as

$$\frac{i(E_k^{n+1}-E_k^n)}{\Delta t}+\frac{1}{2}\delta^2 E_k^n+\frac{1}{2}\delta^2 E_k^{n+1} = \frac{1}{4}(N_k^n+N_k^{n+1})(E_k^n+E_k^{n+1}) \tag{9}$$

$$\frac{N_k^{n+1}-2N_k^n+N_k^{n-1}}{\Delta t^2}-\frac{1}{2}\delta^2 N_k^{n+1} -\frac{1}{2}\delta^2 N_k^{n-1}=\delta^2(|E_k^n|^2). \tag{10}$$

In both expressions $k=1, 2, ..., J$; $n \geqslant 0$ in (9) while $n \geqslant 1$ in (10). E_k^n, N_k^n are to be J-periodic mesh functions, i.e.,

$$E_k^n=E_j^n; \qquad N_k^n=N_j^n \qquad \text{if} \quad k\equiv j \pmod{J}. \tag{11}$$

The scheme is supplemented with the initial values

$$E_k^0=E^0(x_k) \tag{12}$$

$$N_k^0=N^0(x_k); \qquad N_k^1=N_k^0+\Delta t\, N^{-1}(x_k). \tag{13}$$

One begins by putting $n=0$ in (9) and solving for $\{E_k^1\}$ by using the data (12), (13). This involves the solution of a "periodic tridiagonal system" (cf. [7]). Then one puts $n=1$ in (10) and solves for $\{N_k^2\}$; here another such linear system arises. These systems are solved by a threefold application of "standard" tridiagonal solvers, as is described in [7]. This entire process is now repeated to generate $\{E_k^2\}$, $\{N_k^3\}$, etc.

In order to describe the norm in which convergence takes place, we define the "discrete potential" $\{u_k^n\}$ by

$$\delta^2 u_k^n=\frac{N_k^{n+1}-N_k^n}{\Delta t} \qquad (k=1, ..., J-1) \tag{14}$$

with the boundary conditions

$$u_0^n=u_J^n=0 \tag{15}$$

and the periodic extension

$$u_k^n=u_j^n \qquad \text{if} \quad k\equiv j \pmod{J}. \tag{16}$$

Thus u_k^n can be represented as

$$u_k^n=-\Delta x\sum_{j=1}^{J-1} G(x_k, x_j)\frac{N_j^{n+1}-N_j^n}{\Delta t}, \tag{17}$$

where

$$G(x, y)=\begin{cases} x(1-y/L), & 0\leqslant x\leqslant y\leqslant L \\ y(1-x/L), & 0\leqslant y\leqslant x\leqslant L. \end{cases} \tag{18}$$

A "compatibility condition" for definition (14) is, in view of (10), the initial conditions (12), (13), and periodicity, that

$$\sum_{j=1}^{J} N^1(j\,\Delta x)=0. \tag{19}$$

From [3] we have these invariants:

THEOREM 1. *Under the assumptions above, the solution* $\{E_k^n\}$, $\{N_k^n\}$ *of the difference scheme* (9), (10) *satisfies*

(i) $\sum_{k=1}^J |E_k^n|^2\,\Delta x=\text{const}$.

(ii) $\sum_{k=1}^J \Delta x[|\delta E_k^{n+1}|^2+\frac{1}{2}(\delta u_k^n)^2+\frac{1}{4}((N_k^n)^2+(N_k^{n+1})^2) +\frac{1}{2}(N_k^n+N_k^{n+1})\,|E_k^{n+1}|^2]=\text{const}$.

These correspond to the "continuous invariants" (2), (3) and are established by elementary but tedious summations by parts. It can be shown [3] that the discrete energy in (ii) is positive. In fact, from (ii) we can show that

$$\sum_{k=1}^{J} \Delta x\,[|E_k^{n+1}|^2+|\delta E_k^{n+1}|^2+(\delta u_k^n)^2 +(N_k^n)^2+(N_k^{n+1})^2]\leqslant \text{const}. \tag{20}$$

In terms of the exact solution (E, N) of (ZS), we define the *errors* by

$$e_k^n=E(x_k, t^n)-E_k^n \tag{21}$$

$$\eta_k^n=N(x_k, t^n)-N_k^n, \tag{22}$$

where $\{E_k^n\}$, $\{N_k^n\}$ are computed from the scheme (9), (10) for $k=1, ..., J$; $n\,\Delta t\leqslant T$. By analogy to (14), (17) we define $\{U_k^n\}$ by

$$U_k^n=-\Delta x\sum_{j=1}^{J-1} G(x_k, x_j)\frac{\eta_j^{n+1}-\eta_j^n}{\Delta t} \qquad (k=1, ..., J-1) \tag{23}$$

with $U_0^n=U_J^n=0$ and the obvious periodic extension. The convergence theorem from [3] can be stated as follows:

THEOREM 2. *Define the norms*

$$\|e^n\|_2^2=\sum_{k=1}^{J}\Delta x\,|e_k^n|^2 \tag{24}$$

$$\|\delta e^n\|_2^2=\sum_{k=1}^{J}\Delta x\,|\delta e_k^n|^2 \tag{25}$$

etc. Then under the above assumptions we have for $\Delta t=\Delta x$ *sufficiently small the bound*

$$\varepsilon^n\leqslant c_T\,\Delta t$$

for $n\,\Delta t \leqslant T$, where ε^n, the square of the "energy norm," is defined by

$$\varepsilon^n = \|e^{n+1}\|_2^2 + \|\delta e^{n+1}\|_2^2 + \|\delta U^n\|_2^2 + \tfrac{1}{2}(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2). \tag{26}$$

III. THE FORM OF THE SOLITARY WAVES

For the purpose of comparison we will use the notation of [6]. One seeks a solution to (ZS) in the form

$$E(x, t) = F(x - vt)\, e^{i\phi(x - ut)} \tag{27}$$

$$N(x, t) = G(x - vt). \tag{28}$$

Here v, ϕ, u are real constants with $|v| < 1$. F, G are L-periodic functions of one real variable $\xi = x - vt$. Substituting into (ZS.N) we obtain

$$v^2 G'' - G'' = (|F(\xi)|^2)'' \tag{29}$$

and, hence,

$$G(\xi) = \frac{|F(\xi)|^2}{v^2 - 1} + c_0 + c_1 \xi. \tag{30}$$

By periodicity, $c_1 = 0$. We choose c_0 so that

$$\int_0^L N(x, t)\, dx = 0.$$

Hence,

$$c_0 = \frac{1}{L(1 - v^2)} \int_0^L |F(y)|^2\, dy. \tag{31}$$

Since $N_t(x, t) = -vG'(\xi)$, we have

$$N_t(x, 0) \equiv N^1(x) = -vG'(x) = \frac{-2v}{v^2 - 1} F(x)\, F'(x). \tag{32}$$

Thus the compatibility condition $\int_0^L N^1(x)\, dx = 0$ holds automatically, since $F(\cdot)$ is L-periodic.

The equation for $F(\xi)$ which results from substitution into (ZS.E) is

$$F''(\xi) = \alpha F - \beta F^3, \tag{33}$$

where

$$\alpha = \frac{v^2}{4} - \frac{uv}{2} + c_0; \qquad \beta = \frac{1}{1 - v^2}. \tag{34}$$

In order to obtain this we eliminated the imaginary coefficient of F' by choosing

$$\phi = \frac{v}{2}. \tag{35}$$

A first integral of this is

$$(F')^2 = \alpha F^2 - \frac{\beta}{2} F^4 + \tilde{C}$$

for some constant $\tilde{C}$. Scaling now by $\eta = \sqrt{\beta/2}\, \xi$ we obtain

$$\left(\frac{dF}{d\eta}\right)^2 = -F^4 + \frac{2\alpha}{\beta} F^2 + \frac{2\tilde{C}}{\beta}. \tag{36}$$

Now we choose $\tilde{C}$ so that the right side of (36) can be expressed in the form

$$(1 - F^2)(F^2 - k'^2) \tag{37}$$

for an appropriate constant k'. A brief calculation shows that the choices

$$\tilde{C} = \frac{\beta}{2} - \alpha; \qquad k'^2 = \frac{-2\tilde{C}}{\beta} \tag{38}$$

give us (37). Then we have a standard differential equation

$$(F'(\eta))^2 = (1 - F^2)(F^2 - k'^2)$$

from which it follows that a periodic solution of (33) is given by

$$F(\xi) = dn\left(\frac{\xi}{\sqrt{2(1 - v^2)}}, k\right). \tag{39}$$

Here $dn(\cdot)$ denotes a Jacobian elliptic function (cf. [4, 9]), and

$$k^2 + k'^2 = 1. \tag{40}$$

Solutions with different amplitudes are also possible [6]. The choice (38) now determines u:

$$u = \frac{v}{2} + \frac{2c_0}{v} - \frac{(1 + k'^2)}{v(1 - v^2)}. \tag{41}$$

In view of (35), $\phi = v/2$, the exponential in the ansatz (27) will be L-periodic provided

$$\frac{vL}{2} = 2\pi m \qquad \text{for some} \quad m = 1, 2, \ldots.$$

Below, we will use $m=1$ so that

$$v=4\pi/L. \tag{42}$$

Therefore we will choose periods $L>4\pi$.

Finally we enforce the periodicity of F. One knows that the function

$$u \mapsto dn(u, k)$$

is $2K$-periodic, where

$$K=\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}$$

(cf. [4, 9]). Since $F(\xi)=dn(\xi/\sqrt{2(1-v^2)}, k)$ is to be L-periodic, we are led to the relation

$$L=2\sqrt{2(1-v^2)}\,K \tag{43}$$

which will guarantee periodicity. Incidentally, the last equation is an interesting type of "inverse problem." Since L is given and v is known from (42), we need to find k so that (43) holds. We achieve this using educated guesses and a result from [1, p. 591]: for the function

$$K(m)\equiv\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m \sin^2 \theta}} \qquad (0\leqslant m<1),$$

one has for appropriate numerical values $a_0, ..., b_2$ the approximation

$$K(m)\equiv a_0+a_1 m_1+a_2 m_1^2 + (b_0+b_1 m_1+b_2 m_1^2)\ln\left(\frac{1}{m_1}\right)+\varepsilon(m), \tag{44}$$

where $m+m_1=1$ and $|\varepsilon(m)|\leqslant 3\cdot 10^{-5}$.

From (41) u is determined, and all the parameters will be known, once c_0 is computed. For this we have from (31)

$$c_0=\frac{1}{L(1-v^2)}\int_0^L dn^2\left(\frac{\xi}{\sqrt{2(1-v^2)}}, k\right)d\xi = \frac{\sqrt{2(1-v^2)}}{L(1-v^2)}\int_0^{L/\sqrt{2(1-v^2)}} dn^2(u, k)\,du. \tag{45}$$

From (43) the upper limit here equals $2K$. By symmetry of $dn(\cdot, k)$ then and by [9, p. 518], we find

$$c_0=\frac{\sqrt{2}}{L\sqrt{1-v^2}}\cdot 2\cdot\int_0^{\pi/2}\sqrt{1-k^2\sin^2\phi}\,d\phi. \tag{46}$$

This completes the structural computation of the solitary waves.

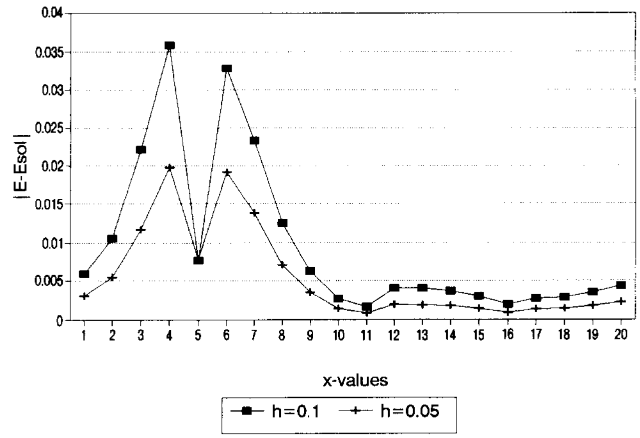

FIG. 1. $|E-E_{\mathrm{sol}}|$, $L=20$, $t=8$.

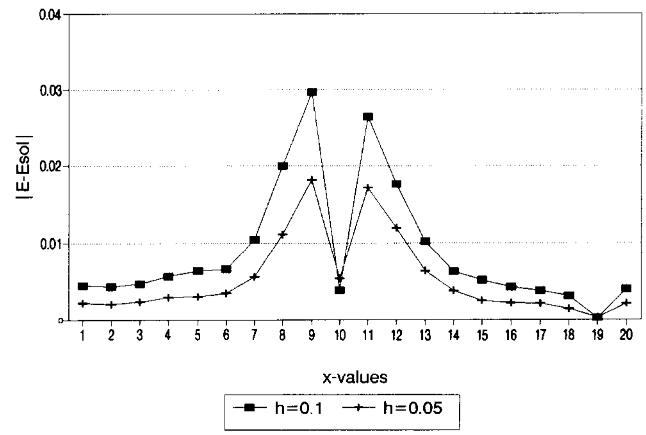

FIG. 2. $|E-E_{\mathrm{sol}}|$, $L=20$, $t=16$.

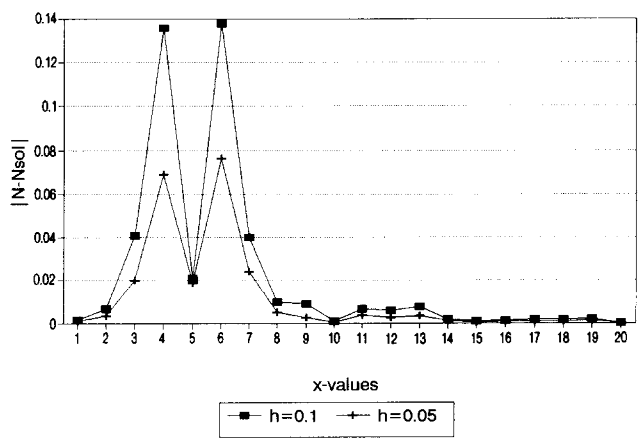

FIG. 3. $|N-N_{\mathrm{sol}}|$, $L=20$, $t=8$.

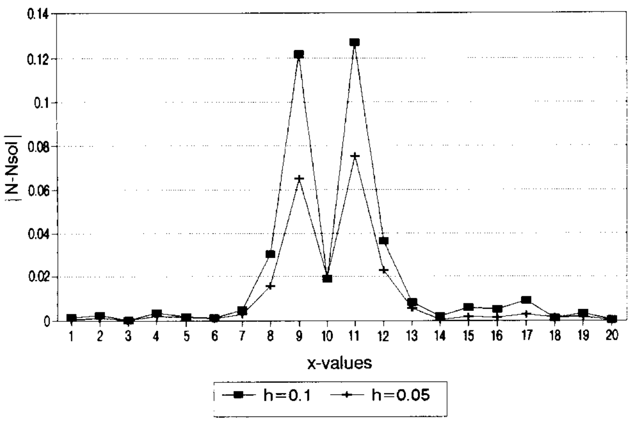

FIG. 4. $|N - N_{\mathrm{sol}}|$, $L = 20$, $t = 16$.

IV. COMPUTATION OF SOLITARY WAVES

We ran the difference method (9), (10) with the following parameters (chosen and verified from [6]): $L = 20$, $v = 4\pi/L = 0.6283185$; $k' = 4.5147 \cdot 10^{-4}$, $K = 9.089296$ (using (43) and (44)); $u = -1.73692$ (from (41)), $c_0 = 0.181786$ (from (46)). We made two runs with $h = \Delta t = \Delta x = 0.1$ and $h = 0.05$. For comparison, we computed the solitary wave solution (called E_{sol}, N_{sol} in the figures). The figures show the absolute value of the errors $|E - E_{\mathrm{sol}}|$, $|N - N_{\mathrm{sol}}|$ at two real times 8 and 16 as functions of x, $0 \leqslant x \leqslant 20$. (Of course E, N here denote the solution of the scheme (9), (10).) As is seen, cutting the step size in half roughly cuts the error in half, as expected. The maximum

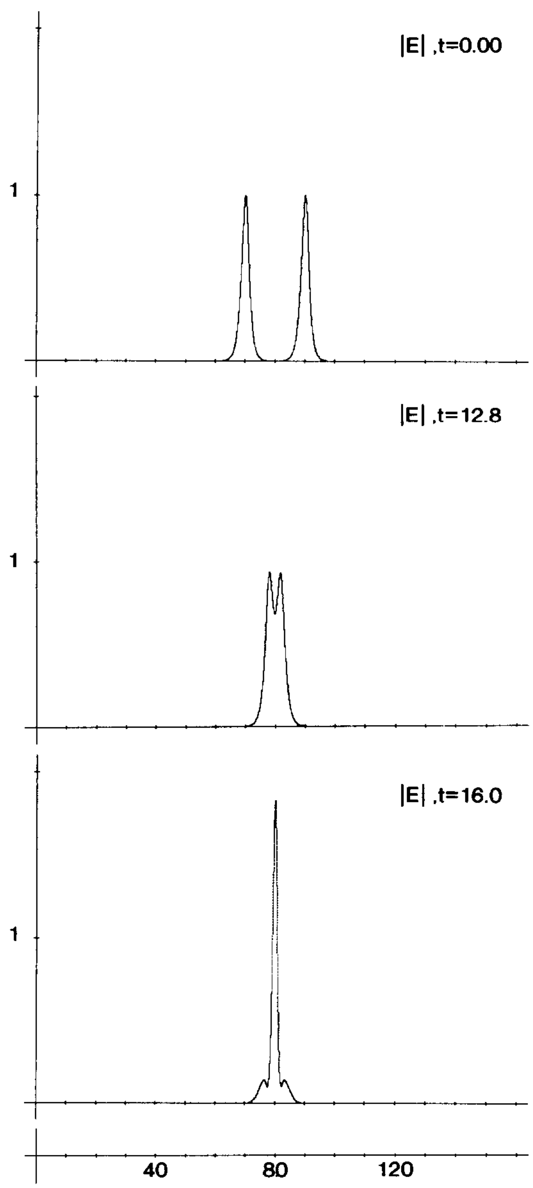

FIG. 5. $|E|$ during collision, $t = 0$, $t = 12.8$, $t = 16.0$.

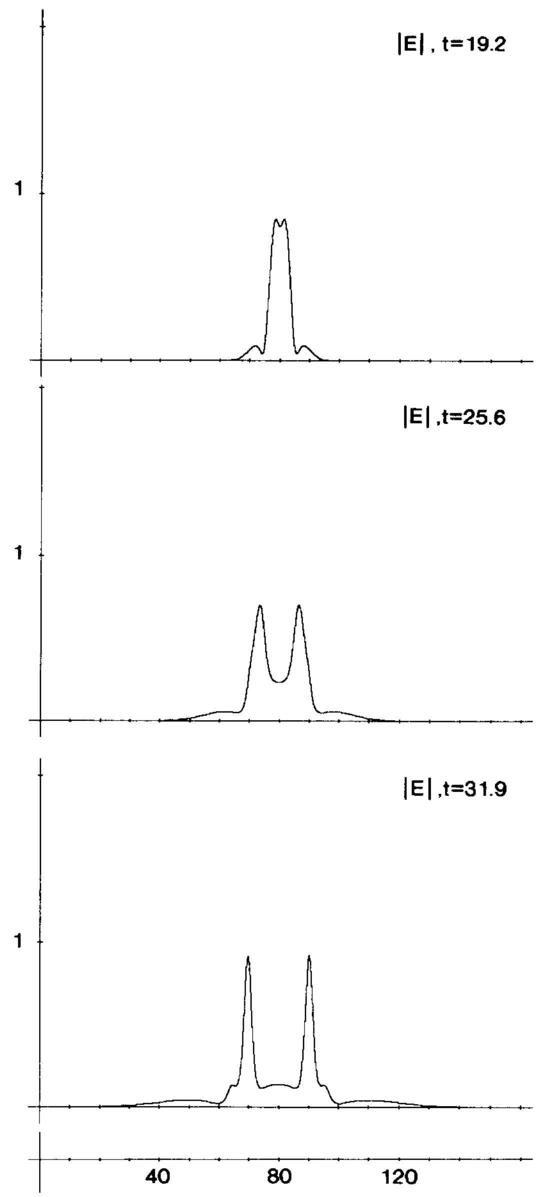

FIG. 6. $|E|$ during collision, $t = 19.2$, $t = 25.6$, $t = 31.9$.

amplitude of $|E_{sol}|$ is max $|F| = 1$; from (30) we obtain crudely that N_{sol} satisfies the bounds $-1.6523 = 1/(v^2 - 1) < N_{sol}(x, t) \leqslant c_0 < 0.2$.

The initial values for E, N are clear from Section III. As for the time derivative N_t, we have (32) for which we need the fact that

$$dn'(u, k) = -k^2 sn(u, k)\, cn(u, k)$$

in standard notation ([4]).

The Collision of Two Solitary Waves

Here we describe the results of our re-doing the computational experiment performed in [6]. On an interval $0 \leqslant x \leqslant L \equiv 160$ we take as initial values two solitons (of period 20, with parameters as in the preceding section) with oppositely-directed velocities. The right-moving soliton is centered at $x = 70$; the left-moving soliton at $x = 90$. By (46) with $L = 160$, we obtain $c_0 = 0.02272323$. These initial values generate the graphs shown in [6, p. 493, 494.]

We ran the experiment twice, once with $h = \Delta t = \Delta x = 0.1$ and again with $h = 0.05$. In the figures we display for $h = 0.05$ both $|E|$ and N at various (real) times as a function of x, $0 \leqslant x \leqslant L = 160$. Just before the interaction one has the picture shown at time 12.8. The solitons roughly coincide at time $t = 16$; the final graphs depict the behavior after the interaction is complete (at approximately $t = 31.8$). The

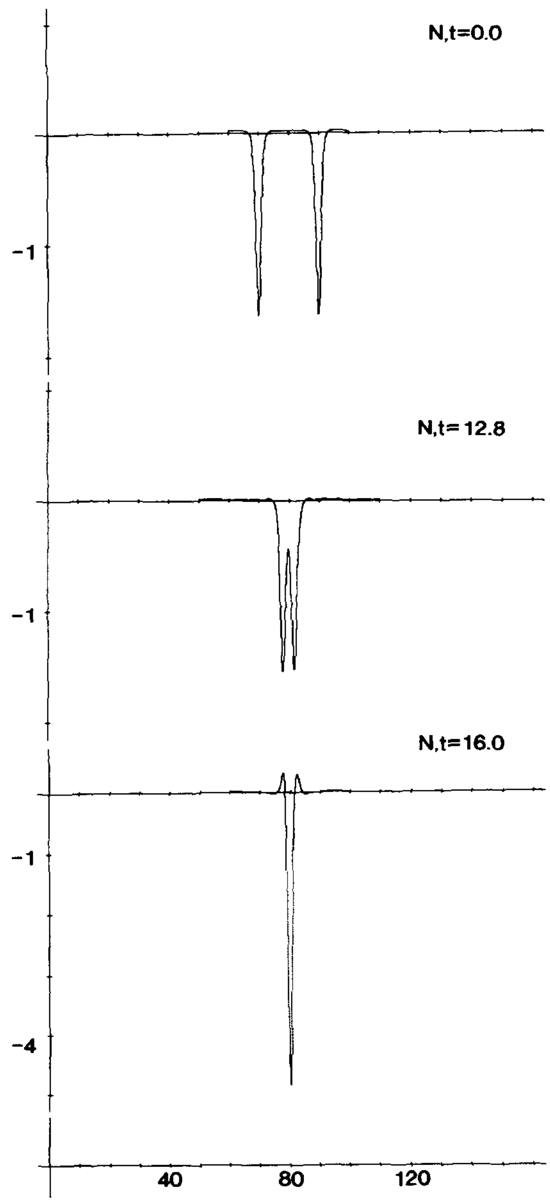

FIG. 7. N during collision, $t = 0$, $t = 12.8$, $t = 16.0$.

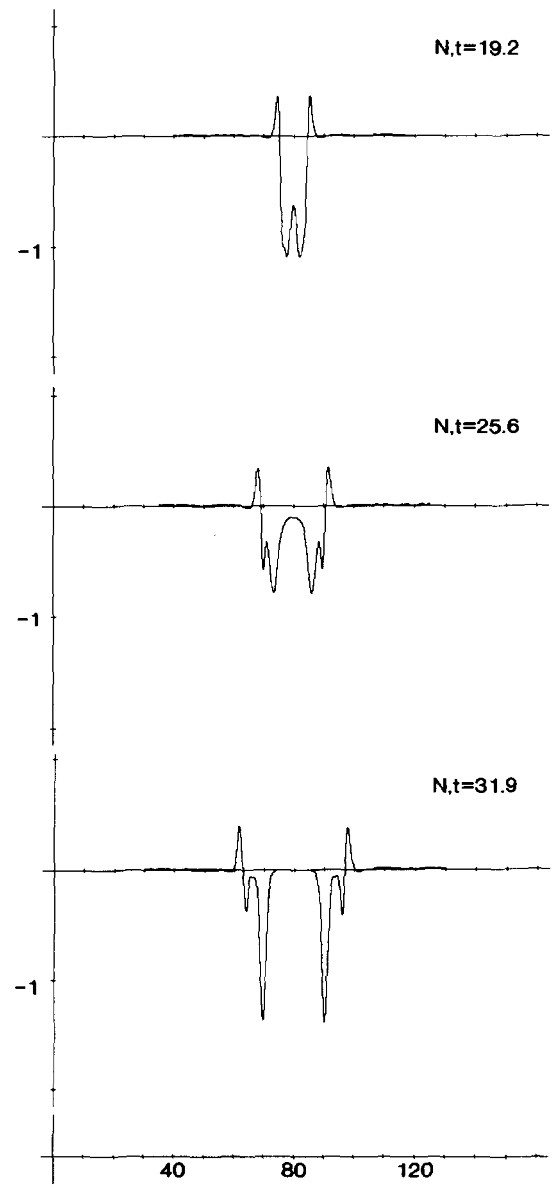

FIG. 8. N during collision, $t = 19.2$, $t = 25.6$, $t = 31.9$.

values of the conserved discrete energy ε_d (from part (ii) of Theorem 1) are computed to be

$$\varepsilon_d = 2.3339714 \qquad (h = 0.1)$$

$$\varepsilon_d = 2.3307398 \qquad (h = 0.05)$$

and remain the same at each time step to as many places as shown.

Comparison of our graphical results with those of [6] shows excellent qualitative agreement. Since the present finite-difference method is known to converge, we expect there is a theorem possible for the spectral method in [6].

In conclusion, the finite-difference method presented here generates output consistent with that of the spectral scheme given in [6]. The scheme conserves the two standard invariants and has been proven to converge.

Similar computations could be attempted in three space dimensions, where it is unknown if finite-time "blowup" can occur. In this case the energy can be negative, suggesting the possibility of singular behavior.

The computations were done on a Sun Sparc Station 1$^+$ and on an Alliant FX/8; the C-code was compiled with gcc.

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